

Vanishing results for the cohomology of complex toric hyperplane complements

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Abstract

Suppose \mathcal{R} is the complement of an essential arrangement of toric hyperplanes in the complex torus $(\mathbb{C}^*)^n$ and $\pi = \pi_1(\mathcal{R})$. We show that $H^*(\mathcal{R}; A)$ vanishes except in the top degree n when A is one of the following systems of local coefficients: (a) a system of nonresonant coefficients in a complex line bundle, (b) the von Neumann algebra $\mathcal{N}\pi$, or (c) the group ring $\mathbb{Z}\pi$. In case (a) the dimension of H^n is $|e(\mathcal{R})|$ where $e(\mathcal{R})$ denotes the Euler characteristic, and in case (b) the n^{th} ℓ^2 Betti number is also $|e(\mathcal{R})|$.

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1 Introduction

A *complex toric arrangement* is a family of complex subtori of a complex torus $(\mathbb{C}^*)^n$. The study of such objects is a relatively recent topic. Different versions of these arrangements, also known as toral arrangements, have been introduced and studied in works of Lehrer [16, 17], Dimca-Lehrer [11], Douglass [12], Looijenga [18] and Macmeikan [20, 21].

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The foundation of the topic can be traced to the paper [10] by De Concini and Procesi. There the main objects are defined and the cohomology of the complement of a toric arrangement is studied. An explicit goal of [10] is to generalize the theory of hyperplane arrangements. (For an extensive account of the work of De Concini and Procesi see [9].)

The next step is the work of Moci, in particular his papers [22, 23, 24], developing the theory with a special focus on combinatorics. In [25] Moci and the second author study the homotopy type of the complement of a special class of toric arrangements which they call *thick*. In [3] D’Antonio and Delucchi generalize results in [25] to a wider class of toric arrangements which they call *complexified* because of structural affinity with the case of hyperplane arrangements. They also prove that complements of complexified toric arrangements are minimal (see [4]).

In this paper we generalize to toric arrangements a well known result for affine arrangements: vanishing conditions for the cohomology of the complement $M(\mathcal{A})$ of an arrangement \mathcal{A} with coefficients in a complex local system A . Necessary conditions for $H^k(M(\mathcal{A}); A) = 0$ if $k \neq n$, i.e., for the cohomology to be concentrated in top dimension, have been determined by a number of authors, including Kohno [15], Esnault, Schechtman and Viehweg [13], Davis, Januszkiewicz and Leary [5], Schechtman, Terao and Varchenko [28] and Cohen and Orlik [2]. In particular, in [28] (see also [2]) it is proved that the cohomology of the complement $M(\mathcal{A})$ of an arrangement with coefficients in a complex local system is concentrated in top dimension provided certain *nonresonance* conditions for monodromies are fulfilled for a certain subset of edges (i.e., intersections of hyperplanes) that are called *denses*.

In order to generalize the above results we use techniques developed by the first author in a joint work with Januszkiewicz, Leary and Okun, [5, 6, 7, 8]. One considers an open cover of the complement M by “small” open sets each homeomorphic to the complement of a central arrangement. In the cases of nonresonant rank one local coefficients or ℓ^2 coefficients, the E_1 page of the resulting Mayer-Vietoris spectral sequence is nonzero only along the bottom row, where it can be identified with the simplicial cochains with constant coefficients on a pair $(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$, which is homotopy equivalent to (\mathbb{C}^n, Σ) where Σ is the union of all hyperplanes in the arrangement. (The simplicial complex $N(\mathcal{U})$ is the nerve of an open cover of \mathbb{C}^n and $N(\mathcal{U}_{\text{sing}})$ is a subcomplex.)

It follows that the E_2 page can be nonzero only in position $(l, 0)$. One also can prove that for an affine hyperplane arrangement of rank l only the

l^{th} ℓ^2 -Betti number of the complement M can be nonzero and that it is equal to the rank of the reduced $(l - 1)$ -homology of Σ (cf. [5]). Similarly, with coefficients in the group ring, $\mathbb{Z}\pi$, for $\pi = \pi_1(M)$, $H^*(M; \mathbb{Z}\pi)$ is nonzero only in degree l (cf. [6]). We generalize all three of these vanishing results to the toric case in Theorems 5.1, 5.2 and 5.3.

In recent work [27], Papadima and Suciu generalize the result in [2] to arbitrary minimal CW-complex, i.e., a complex having as many k -cells as the k -th Betti number. It would be very interesting to decide if the complement of toric arrangement also could be minimal. In this case Theorem 5.1 would be a consequence of minimality.

Our paper begins with a review of some background about toric and affine arrangements. Then, in Section 3, we give a brief account of open covers by “small” convex sets. In Section 4 we recall basic definitions on systems of local coefficients. Finally in Section 5 we prove that the cohomology of the complement of a toric arrangement with coefficient in a local system A vanishes except in the top degree when A is a nonresonant local system, the von Neumann algebra $\mathcal{N}\pi$ or the group ring $\mathbb{Z}\pi$.

2 Affine and toric hyperplane arrangements

Affine hyperplanes arrangements A *hyperplane arrangement* \mathcal{A} is a finite collection of affine hyperplanes in \mathbb{C}^n . A *subspace* of \mathcal{A} is a nonempty intersection of hyperplanes in \mathcal{A} . Denote by $L(\mathcal{A})$ the poset of subspaces, partially ordered by inclusion, and let $\bar{L}(\mathcal{A}) := L(\mathcal{A}) \cup \{\mathbb{C}^n\}$. An arrangement is *central* if $L(\mathcal{A})$ has a minimum element. Given $G \in L(\mathcal{A})$, its *rank*, $\text{rk}(G)$, is the codimension of G in \mathbb{C}^n . The minimal elements of $L(\mathcal{A})$ form a family of parallel subspaces and they all have the same rank. The *rank* of an arrangement \mathcal{A} is the rank of a minimal element in $L(\mathcal{A})$. \mathcal{A} is *essential* if $\text{rk}(\mathcal{A}) = n$.

The *singular set* $\Sigma(\mathcal{A})$ of \mathcal{A} is the union of hyperplanes in \mathcal{A} . The complement of $\Sigma(\mathcal{A})$ in \mathbb{C}^n is denoted $M(\mathcal{A})$.

Toric arrangements Let $T = (\mathbb{C}^*)^n$ be a complex torus and let $\Lambda = \text{Hom}(T, \mathbb{C}^*)$ denote the group of characters of T . Then $\Lambda \cong \mathbb{Z}^n$. A character is *primitive* if it is a primitive vector in Λ . Given a primitive character χ and an element $a \in \mathbb{C}^*$ put

$$H_{\chi,a} = \{t \in T \mid \chi(t) = a\}.$$

The subtorus $H_{\chi,a}$ is a *toric hyperplane*. A finite subset $X \subset \Lambda \times \mathbb{C}^*$ defines a *toric arrangement*,

$$\mathcal{T}_X := \{H_{\chi,a}\}_{(\chi,a) \in X}$$

The projection of X onto the first factor is denoted $p(X)$ and is called the *character set* of \mathcal{T}_X . (Thus, $p(X) := \{\chi \mid (\chi, a) \in X\}$.) The *singular set*, Σ_X , is the union of toric hyperplanes in the arrangement. Its complement, $T - \Sigma_X$, is denoted \mathcal{R}_X . The *intersection poset* L_X is the set of nonempty intersections of toric hyperplanes and $\bar{L}_X = L_X \cup \{T\}$. \bar{L}_X is partially ordered by inclusion. The *rank* of the arrangement is the dimension of the linear subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ spanned by $p(X)$. The arrangement is *essential* if its rank is n .

Suppose $G \in L_X$. Choose a point $x \in G$. The *tangential arrangement along G* is the arrangement \mathcal{A}_G of linear hyperplanes which are tangent to the complex toric hyperplanes containing G (i.e., all hyperplanes of the form $T_x(H_{\chi,a})$ where $T_x(G) \subset T_x(H_{\chi,a})$). It is a central hyperplane arrangement of rank equal to $n - \dim G$.

Given a toric arrangement \mathcal{T}_X of rank l , let K_X denote the identity component of the intersection of all kernels in $p(X)$, i.e., K_X is the identity component of

$$\bigcap_{\chi \in p(X)} \text{Ker } \chi = \{t \in T \mid \chi(t) = 1, \forall \chi \in p(X)\}.$$

Put $\bar{T}_X := T/K_X$. Thus, K_X and \bar{T}_X are tori of dimensions $n - l$ and l , respectively. ($K_X \cong (\mathbb{C}^*)^{n-l}$ and $\bar{T}_X \cong (\mathbb{C}^*)^l$.) Let $\bar{\Sigma}_X$ denote the image of Σ_X in \bar{T}_X . Since $T \rightarrow T/K_X$ is a trivial K_X -bundle, we have a homeomorphism of pairs,

$$(T, \Sigma_X) \cong K_X \times (\bar{T}_X, \bar{\Sigma}_X). \quad (1)$$

In other words, the arrangement in T is just the product of the arrangement in \bar{T}_X with the torus K_X . We call $\bar{\mathcal{T}}_X$ the *essentialization* of \mathcal{T}_X . So, it is not restrictive to consider essential toric arrangements.

Lemma 2.1. (cf. [5, Prop. 2.1]). *Suppose \mathcal{T}_X is an essential toric arrangement on T and $\Sigma = \Sigma_X$. Then $H_*(T, \Sigma)$ is free abelian and concentrated in degree n .*

Proof. We follow the “deletion-restriction” argument in [5, Prop. 2.1]) using induction on $\text{Card}(\mathcal{T}_X)$. Choose a toric hyperplane $H \in \mathcal{T}_X$. Let $\mathcal{T}' = \mathcal{T}_X -$

$\{H\}$ and let \mathcal{T}'' be the restriction of \mathcal{T}_X to H , i.e., $\mathcal{T}'' = \{H \cap H' \mid H' \in \mathcal{T}_X\}$. Let Σ' and Σ'' denote the singular sets of \mathcal{T}' and \mathcal{T}'' , respectively. Consider the exact sequence of the triple (T, Σ, Σ') ,

$$\rightarrow H_*(T, \Sigma') \rightarrow H_*(T, \Sigma) \rightarrow H_{*-1}(\Sigma, \Sigma') \rightarrow \quad (2)$$

There is an excision, $H_{*-1}(\Sigma, \Sigma') \cong H_{*-1}(H, \Sigma'')$. The rank of \mathcal{T}' is either n or $n - 1$, while the rank of \mathcal{T}'' is always $n - 1$. The argument breaks into two cases depending on the rank of \mathcal{T}' .

Case 1: the rank of \mathcal{T}' is n . By induction, $H_*(T, \Sigma')$ and $H_*(H, H \cap \Sigma)$ are free abelian and concentrated in degrees n and $n - 1$, respectively. So, (2) becomes,

$$0 \rightarrow H_n(T, \Sigma') \rightarrow H_n(T, \Sigma) \rightarrow H_{n-1}(H, H \cap \Sigma') \rightarrow 0$$

and all other terms are 0. Therefore, $H_*(T, \Sigma)$ is concentrated in degree n and $H_n(T, \Sigma)$ is free abelian.

Case 2: the rank of \mathcal{T}' is $n - 1$. Then the projection $T \rightarrow \overline{T}$ takes H isomorphically onto \overline{T} and the arrangement \mathcal{T}'' on H maps isomorphically to the arrangement $\overline{\mathcal{T}}_X$ on \overline{T} . So, $(H, H \cap \Sigma') \cong (\overline{T}, \overline{\Sigma})$. By (1), $(T, \Sigma') \cong K_X \times (H, H \cap \Sigma') \cong \mathbb{C}^* \times (H, H \cap \Sigma')$. By the Künneth Formula, $H_*(T, \Sigma') \cong H_*(\mathbb{C}^*) \otimes H_*(H, H \cap \Sigma')$. So,

$$\begin{aligned} H_{n-1}(T, \Sigma') &\cong H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \quad \text{and} \\ H_n(T, \Sigma') &\cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma'); \end{aligned}$$

moreover, the first isomorphism is induced by the inclusion $(H, H \cap \Sigma') \rightarrow (T, \Sigma')$. So, (2) becomes,

$$\begin{aligned} 0 \rightarrow H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') &\rightarrow H_n(T, \Sigma) \rightarrow H_{n-1}(H, H \cap \Sigma') \\ &\rightarrow H_0(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma') \end{aligned}$$

where the last map is an isomorphism. It follows that $H_{n-1}(T, \Sigma) = 0$ and that $H_n(T, \Sigma) \cong H_1(\mathbb{C}^*) \otimes H_{n-1}(H, H \cap \Sigma')$, which, by inductive hypothesis, is free abelian. This proves the lemma. \square

Complexified toric arrangements In [3] D'Antonio-Delucchi consider the case of “complexified toric arrangements.” This means that for each $(\chi, a) \in X$, the complex number a has modulus 1 (where $X \subset \Lambda \times \mathbb{C}^*$ is a

set defining a toric arrangement \mathcal{T}_X). Let $T^{\text{cpt}} = (S^1)^n \subset \mathbb{C}^n$ be the compact torus. Then for each $H \in \mathcal{T}_X$, $H \cap T^{\text{cpt}}$ is a compact subtorus of T^{cpt} . The set of subtori, $\mathcal{T}_X^{\text{cpt}} := \{H \cap \mathcal{S} \mid H \in \mathcal{T}\}$, is called the associated *compact arrangement*.

Let $\Sigma^{\text{cpt}} := \Sigma_X \cap T^{\text{cpt}}$. We note that (T, Σ_X) deformation retracts onto $(T^{\text{cpt}}, \Sigma^{\text{cpt}})$. Here are a few observations.

- (i) The universal cover of T^{cpt} is \mathbb{R}^n (actually the subspace $i\mathbb{R}^n \subset \mathbb{C}^n$). Let $\pi : \mathbb{R}^n \rightarrow T^{\text{cpt}}$ be the covering projection. Then for each $H^{\text{cpt}} \in \mathcal{T}_X^{\text{cpt}}$, each component of $\pi^{-1}(H^{\text{cpt}})$ is an affine hyperplane and the collection of these hyperplanes is a periodic affine hyperplane arrangement in \mathbb{R}^n .
- (ii) If \mathcal{T}_X is essential, then Σ^{cpt} cuts T^{cpt} into a disjoint union of convex polytopes, called *chambers* (see [25]). The inverse images of these polytopes under π give a tiling of \mathbb{R}^n .
- (iii) When \mathcal{T}_X is essential, it follows from (ii) that for $n \geq 2$, Σ^{cpt} is connected and that for $n \geq 3$, $\pi_1(\Sigma^{\text{cpt}}) = \pi_1(T^{\text{cpt}})$.
- (iv) It is easy to prove Lemma 2.1 in the case of a compact arrangement. We have an excision $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}}) \cong H_*(\coprod (P_i, \partial P_i))$ where each chamber P_i is an n -dimensional convex polytope. Hence, $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is concentrated in degree n and is free abelian. Moreover, the rank of $H_*(T^{\text{cpt}}, \Sigma^{\text{cpt}})$ is the number of chambers.
- (v) Let $\tilde{\Sigma}^{\text{cpt}}$ denote the inverse image of Σ^{cpt} in \mathbb{R}^n and let $\tilde{\Sigma}_X$ be the induced cover of Σ_X . Suppose \mathcal{T}_X is essential. Then $\tilde{\Sigma}^{\text{cpt}}$ cuts \mathbb{R}^n into compact chambers. It follows that $\tilde{\Sigma}^{\text{cpt}}$ (and hence, $\tilde{\Sigma}$) is homotopy equivalent to a wedge of $(n-1)$ -spheres.

3 Certain covers and their nerves

Equip the torus $T = (\mathbb{C}^*)^n$ with an invariant metric. This lifts to a Euclidean metric on \mathbb{C}^n induced from an inner product. Hence, geodesics in T lift to straight lines in \mathbb{C}^n and each component of the inverse image of a subtorus of T is an affine subspace of \mathbb{C}^n . A *convex subset* of T means a geodesically convex subset. Thus, each component of the inverse image of a convex subset of T is a convex subset of \mathbb{C}^n .

The intersection of an open convex subset of T with the toric hyperplanes in \mathcal{T}_X is equivalent to an affine arrangement. An open convex subset $U \subset T$ is *small* (with respect to \mathcal{T}_X) if this affine arrangement is central. In other words, U is *small* if the following two conditions hold (cf. [5, 6]):

- (i) $\{G \in \overline{L}(\mathcal{T}_X) \mid G \cap U \neq \emptyset\}$ has a unique minimum element, $\text{Min}(U)$.
- (ii) A toric hyperplane $H \in \mathcal{T}_X$ has nonempty intersection with U if and only if $\text{Min}(U) \subset H$.

If (i) and (ii) hold, then the arrangement in U is equivalent to the tangential arrangement along $\text{Min}(U)$, which we denote by $\mathcal{A}_{\text{Min}(U)}$. The intersection of two small convex open sets is also a small convex set; hence, the same is true for any finite intersection of such sets.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of T by small convex sets, put

$$\mathcal{U}_{\text{sing}} := \{U \in \mathcal{U} \mid U \cap \Sigma_X \neq \emptyset\}.$$

Given a nonempty subset $\sigma \subset I$, put $U_\sigma := \bigcap_{i \in \sigma} U_i$. The *nerve* $N(\mathcal{U})$ of \mathcal{U} is the simplicial complex defined as follows. Its vertex set is I and a finite, nonempty subset $\sigma \subset I$ spans a simplex of $N(\mathcal{U})$ if and only if U_σ is nonempty. We have the following lemma.

Lemma 3.1. *Suppose \mathcal{T}_X is essential. $N(\mathcal{U})$ is homotopy equivalent to T and $N(\mathcal{U}_{\text{sing}})$ is a subcomplex homotopy equivalent to Σ_X . Moreover, $H_*(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is concentrated in degree n and $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$ is free abelian.*

Proof. $\mathcal{U}_{\text{sing}}$ is an open cover of a neighborhood of Σ_X which deformation retracts onto Σ_X . For each simplex σ of $N(\mathcal{U})$, U_σ is contractible (in fact, it is a small convex open set). By a well-known result (see [14, Cor. 4G.3 and Ex. 4G(4)]) $N(\mathcal{U})$ is homotopy equivalent to T and $N(\mathcal{U}_{\text{sing}})$ is homotopy equivalent to Σ_X . The last sentence of the lemma follows from Lemma 2.1. \square

Definition 3.2. $\beta(\mathcal{T}_X)$ is the rank of $H_n(N(\mathcal{U}), N(\mathcal{U}_{\text{sing}}))$.

Equivalently, $\beta(\mathcal{T}_X)$ is the rank of $H_n(T, \Sigma_X)$. It is not difficult to see that, for essential arrangements, $(-1)^n \beta(\mathcal{T}_X) = e(T, \Sigma_X) = -e(\Sigma_X) = e(\mathcal{R}_X)$, where $e(\cdot)$ denotes Euler characteristic.

4 Local coefficients

Generic and nonresonant coefficients Consider an affine arrangement \mathcal{A} . The fundamental group π of its complement, $M(\mathcal{A})$, is generated by loops a_H for $H \in \mathcal{A}$, where the loop a_H goes once around the hyperplane H in the “positive” direction. Let α_H denote the image of a_H in $H_1(M(\mathcal{A}))$. Then $H_1(M(\mathcal{A}))$ is free abelian with basis $\{\alpha_H\}_{H \in \mathcal{A}}$. So, a homomorphism $H_1(M(\mathcal{A})) \rightarrow \mathbb{C}^*$ is determined by an \mathcal{A} -tuple $\Lambda \in (\mathbb{C}^*)^{\mathcal{A}}$, where $\Lambda = (\lambda_H)_{H \in \mathcal{A}}$ corresponds to the homomorphism sending α_H to λ_H . Let $\psi_\Lambda : \pi \rightarrow \mathbb{C}^*$ be the composition of this homomorphism with the abelianization map $\pi \rightarrow H_1(M(\mathcal{A}))$. The resulting rank one local coefficient system on $M(\mathcal{A})$ is denoted A_Λ .

Returning to the case where \mathcal{T}_X is a toric arrangement, for each simplex σ in $N(\hat{\mathcal{U}})$, let $\mathcal{A}_\sigma := \mathcal{A}_{\text{Min}(U_\sigma)}$ be the corresponding central arrangement (so that $\hat{U}_\sigma \cong M(\mathcal{A}_\sigma)$). Given $\Lambda_\sigma \in (\mathbb{C}^*)^{\mathcal{A}_\sigma}$, put

$$\lambda_\sigma := \prod_{H \in \mathcal{A}_\sigma} \lambda_H.$$

Let $A_{\Lambda_T} \in \text{Hom}(H_1(\mathcal{R}_X), \mathbb{C}^*)$ be a local coefficient system on \mathcal{R}_X . The localization of A_{Λ_T} on the open set \hat{U}_σ has the form A_{Λ_σ} , where Λ_σ is a \mathcal{A}_σ -tuple in \mathbb{C}^* . We call Λ_T *generic* if $\lambda_\sigma \neq 1$ for all $\sigma \in N(\mathcal{U}_{\text{sing}})$. We call Λ_T *nonresonant* if Λ_σ is nonresonant in the sense of [2] for all $\sigma \in N(\mathcal{U}_{\text{sing}})$ i.e., if the Betti numbers of $M(\mathcal{A}_\sigma)$ with coefficients in A_{Λ_σ} are minimal.

ℓ^2 -cohomology and coefficients in a group von Neumann algebra

For a discrete group π , $\ell^2\pi$ denotes the Hilbert space of complex-valued, square integrable functions on π . There are unitary π -actions on $\ell^2\pi$ by either left or right multiplication; hence, $\mathbb{C}\pi$ acts either from the left or right as an algebra of operators. The *associated von Neumann algebra* $\mathcal{N}\pi$ is the commutant of $\mathbb{C}\pi$ (acting from, say, the right on $\ell^2\pi$).

Given a finite CW complex Y with fundamental group π , the space of ℓ^2 -cochains on the universal cover \tilde{Y} is equal to $C^*(Y; \ell^2\pi)$, the cochains with local coefficients in $\ell^2\pi$. The image of the coboundary map need not be closed; hence, $H^*(Y; \ell^2\pi)$ need not be a Hilbert space. To remedy this, one defines the *reduced* ℓ^2 -cohomology $H_{\text{red}}^*(Y; \ell^2\pi)$ to be the quotient of the space of cocycles by the closure of the space of coboundaries. The von Neumann algebra admits a trace. Using this, one can attach a “dimension,”

$\dim_{\mathcal{N}\pi} V$, to any closed, π -stable subspace V of a finite direct sum of copies of $\ell^2\pi$ (it is the trace of orthogonal projection onto V). The nonnegative real number $\dim_{\mathcal{N}\pi}(H_{\text{red}}^p(Y; \ell^2\pi))$ is the p^{th} ℓ^2 -Betti number of Y .

A technical advance of Lück [19, Ch. 6] is the use local coefficients in $\mathcal{N}\pi$ in place of the previous version of ℓ^2 -cohomology. He shows there is a well-defined dimension function on $\mathcal{N}\pi$ -modules, $A \rightarrow \dim_{\mathcal{N}\pi} A$, which gives the same answer for ℓ^2 -Betti numbers, i.e., for each p one has that $\dim_{\mathcal{N}\pi} H^p(Y; \mathcal{N}\pi) = \dim_{\mathcal{N}\pi} H_{\text{red}}^p(Y; \ell^2\pi)$.

Group ring coefficients Let Y be a connected CW complex, $\pi = \pi_1(Y)$ and $r : \tilde{Y} \rightarrow Y$ the universal cover. There is a well-defined action of π on \tilde{Y} and hence, on the cellular chain complex of \tilde{Y} . Given the left π -module $\mathbb{Z}\pi$, define the cochain complex with *group ring coefficients*

$$C^*(Y; \mathbb{Z}\pi) := \text{Hom}_{\pi}(C_*(\tilde{Y}), \mathbb{Z}\pi).$$

Taking cohomology gives $H^*(Y; \mathbb{Z}\pi)$.

5 The Mayer-Vietoris spectral sequence

Statements of the main theorems Suppose \mathcal{T}_X is an essential toric arrangement in T and $\pi = \pi_1(\mathcal{R}_X)$.

Theorem 5.1. *Let Λ_T be a generic X -tuple with entries in k^* . Then $H^*(\mathcal{R}_X; A_{\Lambda_T})$ is concentrated in degree n and*

$$\dim_k H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X).$$

Theorem 5.2. (cf. [7]). *The ℓ^2 -Betti numbers of \mathcal{R}_X are 0 except in degree n and $\ell^2 b_n(\mathcal{R}_X) = \beta(\mathcal{T}_X)$.*

Theorem 5.3. (cf. [6, 8]). *$H^*(\mathcal{R}_X; \mathbb{Z}\pi)$ vanishes except in degree n and $H^n(\mathcal{R}_X; \mathbb{Z}\pi)$ is free abelian.*

Remark 5.4. Suppose W is a Euclidean reflection group acting on \mathbb{R}^n and that $\mathbb{Z}^n \subset W$ is the subgroup of translations. The quotient $W' := W/\mathbb{Z}^n$ is a finite Coxeter group. The reflection group W acts on the complexification \mathbb{C}^n and W' acts on the torus $T = \mathbb{C}^n/\mathbb{Z}^n$. The image of the affine reflection arrangement in \mathbb{C}^n gives a toric arrangement \mathcal{T}_X in T . The fundamental

group of \mathcal{R}_X is the Artin group A associated to W and \mathcal{R}_X is the Salvetti complex associated to A . The quotient of the compact torus by W' can be identified with the fundamental simplex Δ of W on \mathbb{R}^n . (If W is irreducible, then Δ is a simplex.) It follows that $\beta(\mathcal{T}_x)$ is the order of W' (i.e., the index of \mathbb{Z}^n in W). So, in this case Theorem 5.2 is a special case of the main result of [7] and Theorem 5.3 is a special case of a result of [8, Thm. 4.1].

Lemma 5.5. *Suppose \mathcal{A} is a finite, central arrangement of affine hyperplanes. Let $\pi' = \pi_1(M(\mathcal{A}))$. Then*

- (i) (cf. [28, 2, 5]). *For any generic system of local coefficients A , $H^*(M(\mathcal{A}); A)$ vanishes in all degrees.*
- (ii) (cf. [5]). *$H^*(M(\mathcal{A}); \mathcal{N}\pi')$ vanishes in all degrees. Hence, all ℓ^2 -Betti numbers are 0.*
- (iii) (cf. [6]). *If the rank of \mathcal{A} is l , then $H^*(M(\mathcal{A}); \mathbb{Z}\pi')$ vanishes except in the top degree, l .*

Proofs using the Mayer-Vietoris spectral sequence The proofs of these three theorems closely follow the argument in [7], [5] and particularly, in [6]. For $\pi = \pi_1(\mathcal{R}_X)$, let A denote one of the left π -modules in Section 4.

Let $\mathcal{U} = \{U_i\}$ be an open cover of T by small convex sets. We may suppose that \mathcal{U} is finite and that it is closed under taking intersections. For each $G \in \overline{\mathcal{L}}_X$, put

$$\begin{aligned}\mathcal{U}_G &:= \{U \in \mathcal{U} \mid \text{Min}(U) \leq G\}, \\ \mathcal{U}_G^{\text{sing}} &:= \{U \in \mathcal{U} \mid \text{Min}(U) < G\} = \{U \in \mathcal{U}_G \mid U \cap \Sigma_X \cap G \neq \emptyset\}.\end{aligned}$$

The open cover \mathcal{U} restricts to an open cover $\widehat{\mathcal{U}} = \{U - \Sigma_X\}_{U \in \mathcal{U}}$ of \mathcal{R}_X . Any element $\widehat{U} = U - \Sigma_X$ of the cover is homotopy equivalent to the complement of a central arrangement $M(\mathcal{A}_{\text{Min}(U)})$.

Suppose $N(\mathcal{U})$ is the nerve of \mathcal{U} and $N(\mathcal{U}_G)$ is the subcomplex defined by \mathcal{U}_G . Since $N(\mathcal{U}_G)$ and $N(\mathcal{U}_G^{\text{sing}})$ are nerves of covers of G and $\Sigma_X \cap G$, respectively, by contractible open subsets, we have that for each $G \in \overline{\mathcal{L}}(\mathcal{A})$,

$$H^*(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}})) = H^*(G, \Sigma(\mathcal{T}_X \cap G)). \quad (3)$$

For each k -simplex $\sigma = \{i_0, \dots, i_k\}$ in $N(\mathcal{U})$, let

$$U_\sigma := U_{i_0} \cap \dots \cap U_{i_k}$$

denote the corresponding intersection.

Let $r : \widetilde{\mathcal{R}}_X \rightarrow \mathcal{R}_X$ be the universal cover. The induced open cover $\{r^{-1}(\widehat{U})\}$ of $\widetilde{\mathcal{R}}_X$ has the same nerve $N(\widehat{\mathcal{U}})$ ($= N(\mathcal{U})$). We have the Mayer–Vietoris double complex,

$$C_{i,j} := \bigoplus_{\sigma \in N^{(i)}} C_j(r^{-1}(\widehat{U}_\sigma)),$$

where $N^{(i)}$ denotes the set of i -simplices in $N(\mathcal{U})$ (cf. [1, Ch. VII].) We get a corresponding double cochain complex,

$$E_0^{i,j} := \text{Hom}_\pi(C_{i,j}, A), \quad (4)$$

where $\pi = \pi_1(\mathcal{R}_X)$. The filtration on the double complex gives a spectral sequence converging to the associated graded module for cohomology:

$$\text{Gr } H^m(\mathcal{R}_X; A) = E_\infty := \bigoplus_{i+j=m} E_\infty^{i,j}.$$

By first using the horizontal differential, there is a spectral sequence with E_1 page

$$E_1^{i,j} = C^i(N(\mathcal{U}); \mathcal{H}^j(A))$$

where $\mathcal{H}^j(A)$ is the coefficient system on $N(\mathcal{U})$ defined by

$$\sigma \mapsto H^j(\widehat{\mathcal{U}}_\sigma; A),$$

where $\widehat{\mathcal{U}}_\sigma \cong M(\mathcal{A}_{\text{Min}(U_\sigma)})$. For $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$ these coefficients are 0 for $G \neq T$. For $A = \mathbb{Z}\pi$, they are 0 for $j \neq \dim(G)$. Hence, in all cases, for any coface σ' of σ , if $G' := \text{Min}(U_{\sigma'}) < G$, the coefficient homomorphism $H^j(M(\mathcal{A}_G); A) \rightarrow H^j(M(\mathcal{A}_{G'}); A)$ is the zero map. Moreover, the E_1 page of the spectral sequence decomposes as a direct sum (cf. [8, Lemma 2.2]). In fact, for a fixed j , by using Lemma 5.5, we see that the $E_1^{i,j}$ term decomposes as

$$E_1^{i,j} = \bigoplus_{G \in \widetilde{L}_X^{n-j}} C^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}}); H^j(M(\mathcal{A}_G); A)),$$

where we have constant coefficients in each summand. Hence, at E_2 we have

$$\begin{aligned} E_2^{i,j} &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(N(\mathcal{U}_G), N(\mathcal{U}_G^{\text{sing}}); H^j(M(\mathcal{A}_G); A)) \\ &= \bigoplus_{G \in \overline{L}_X^{n-j}} H^i(G, \Sigma_X \cap G; H^j(M(\mathcal{A}_G); A)), \end{aligned} \quad (5)$$

where the second equation follows from (3).

When $A = A_{\Lambda_T}$ or $A = \mathcal{N}\pi$, all summands vanish for $G \neq T$ and $j \neq 0$. So, we are left with $E_2^{n,0} = H^n(T, \Sigma_X; A)$, which is isomorphic to the tensor product free abelian group of rank $\beta(\mathcal{T}_X)$ with A . It follows that $H^*(\mathcal{R}_X; A)$ is concentrated in degree n and that $\dim_{\mathbb{C}} H^n(\mathcal{R}_X; A_{\Lambda_T}) = \beta(\mathcal{T}_X) = \dim_{\mathcal{N}\pi} H^n(\mathcal{R}_X; \mathcal{N}\pi)$. This proves Theorems 5.1 and 5.2.

Consider formula (5) for $A = \mathbb{Z}\pi$. By Lemma 2.1, $H^i(G, \Sigma_X \cap G)$ is concentrated in degree $\dim G = n - j$. Hence, $E_2^{i,j}$ is nonzero (and free abelian) only for $i + j = n$. It follows that the spectral sequence degenerates at E_2 , i.e., $E_2 = E_{\infty}$. This proves Theorem 5.3.

Remark 5.6. Let us remark that the statement of Theorem 5.1 holds even if the local system Λ_T is nonresonant or if it verifies the Schechtman, Terao and Varchenko nonresonance conditions in all small open convex sets, i.e. Λ_{σ} verifies the nonresonance conditions in [28] for all $\sigma \in N(\mathcal{U}_{\text{sing}})$. Indeed under these conditions Lemma 5.5 holds.

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